Autonomous Navigation for Flying Robots

Lecture 5.2: Recap on Probability Theory

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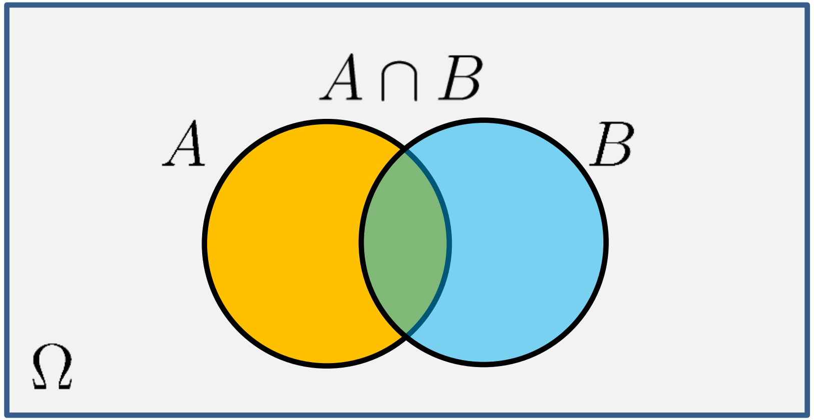
Probability Theory

- Random experiment that can produce a number of outcomes, e.g., rolling a dice
- Sample space, e.g., $\{1, 2, 3, 4, 5, 6\}$
- Event A is subset of outcomes, e.g., $\{2, 4, 6\}$
- Probability $P(A)$, e.g., $P(A) = 0.5$

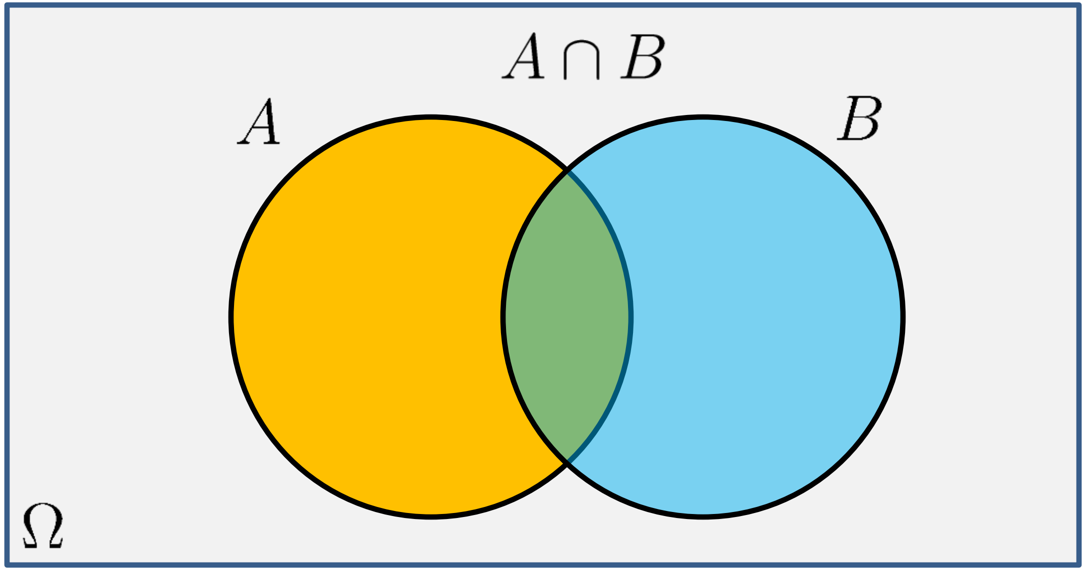
Axioms of Probability Theory

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1 \qquad P(\emptyset) = 0$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



A Closer Look at Axiom 3

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Discrete Random Variables

- X denotes a **random variable**
- X can take on a countable number of values in $\{x_1, x_2, \ldots x_n\}$
- $P(X = x_i)$ is the **probability** that the random variable X takes on value x_i
- $P(\cdot)$ is called the **probability mass function**

- Example: $P(\text{Room}) = < 0.7, 0.2, 0.08, 0.02 >$

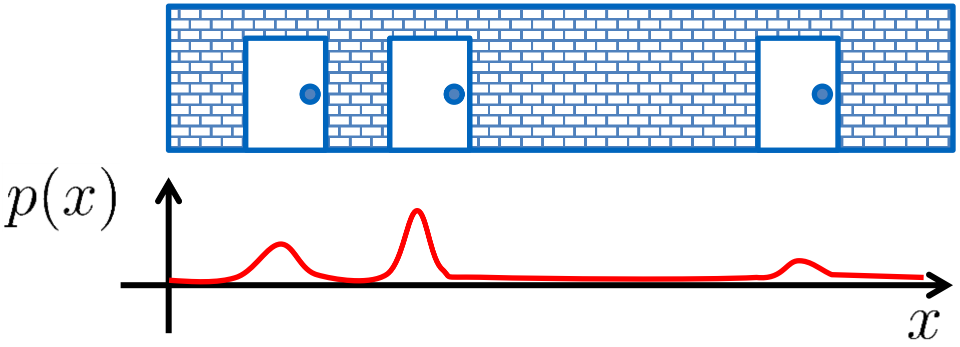
$$\text{Room} \in \{\text{office}, \text{corridor}, \text{lab}, \text{kitchen}\}$$

Continuous Random Variables

- X takes on continuous values
- $p(X = x)$ or $p(x)$ is called the **probability density function (PDF)**

$$P(x \in [a, b]) = \int_a^b p(x)\mathrm{d}x$$

- Example



Thrun, Burgard, Fox: Probabilistic Robotics. MIT Press, 2005
http://mitpress.mit.edu/books/probabilistic-robotics

Proper Distributions Sum To One

- Discrete case $$\sum_x P(x) = 1$$

- Continuous case $$\int p(x)\mathrm{d}x = 1$$

Joint and Conditional Probabilities

- $P(X = x \text{ and } Y = y) = P(x, y)$

- If X and Y are **independent** then
$$P(x, y) = P(x)P(y)$$
- $P(x \mid y)$ is the probability of **x given y**
$$P(x \mid y)P(y) = P(x, y)$$
- If X and Y are independent then
$$P(x \mid y) = P(x)$$

Conditional Independence

- Definition of conditional independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Equivalent to

$$P(x \mid z) = P(x \mid y, z)$$
$$P(y \mid z) = P(y \mid x, z)$$

- Note: this does not necessarily mean that

$$P(x, y) = P(x) P(y)$$

Marginalization

- Discrete case

$$P(x) = \sum_y P(x,y)$$

- Continuous case

$$p(x) = \int p(x,y) \mathrm{d}y$$

Example: Marginalization

P(X,Y)	x_1	x_2	x_3	x_4	P(Y)↓
y_1	1/8	1/16	1/32	1/32	1/4
y_2	1/16	1/8	1/32	1/32	1/4
y_3	1/16	1/16	1/16	1/16	1/4
y_4	1/4	0	0	0	1/4
P(X)→	1/2	1/4	1/8	1/8	1

Expected Value of a Random Variable

- Discrete case $E[X] = \sum_i x_i P(x_i)$
- Continuous case $E[X] = \int x P(X = x) \mathrm{d}x$

- The expected value is the weighted average of all values a random variable can take on.
- Expectation is a linear operator

$$E[aX + b] = aE[X] + b$$

Covariance of a Random Variable

- Measures the squared expected deviation from the mean

$$\mathrm{Cov}[X] = E[X - E[X]]^2 = E[X^2] - E[X]^2$$

Estimation from Data

- Observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$

- Sample Mean $\boldsymbol{\mu} = \frac{1}{n} \sum_i \mathbf{x}_i$

- Sample Covariance

$$\Sigma = \frac{1}{n-1} \sum_i (\mathbf{x}_i - \boldsymbol{\mu})(\mathbf{x}_i - \boldsymbol{\mu})^\top$$

Lessons Learned

- Recap on probability theory
- Random variables
- Joint and conditional probabilities
- Marginalization
- Mean and covariance

- Next:
  Bayes Law and examples